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Fluctuations in quantum and classical populations

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Abstract. Characterization and measurement of fluctuations in quantum and classical populations are compared A simple method for estimating the population fluctuation time is outlined and approximate quantum and classical expressions for the two-time correlation function are derived The utility of the method is demonstrated through analysis of the Scully-Lamb laser model, where it is shown to give correlation times in good agreement with numerical calculations for lasers operated below, at, and above threshold Discrepancies in the predictions of a previous similar method are shown to originate from inconsistent separation of its quantum and classical versions

1. Introduction

Models for the photon statistics of optical sources, whether postulated on a heuristic basis or derived rigorously from fundamental theory, are often couched in the language of population statistics. Indeed, in the absence of field following detectors at optical and higher frequencies of the electromagnetic spectrum, it could be argued that discrete stochastic processes provide the natural theoretical framework for the description of phenomena which can ultimately be probed only by the detection of photoelectric events. There is, of course, a very large body of literature devoted to the analysis of classical population models. These have most commonly been exploited in the biomedical field and in the environmental sciences. However, such models are also used in areas of physics and chemistry In particular, in quantum optics, it is worth noting that the early laser model of Shimoda *et al* (1957) was essentially based on the well known birth-death-immigration process of classical population statistics.

Re-interpreting a classical population model in the context of a quantum mechanical problem is not without pitfalls. In particular the process of measuring or monitoring the population must be handled with care if the formalism associated with the quantization of the Maxwell field is not invoked. This approach has led in the past to some discrepancies appearing in a simple but potentially useful method for estimating the timescale of fluctuations generated by quantum population models (Jakeman and Pike 1971). The purpose of this paper is to clarify the origin of these discrepancies and to demonstrate the utility of the method through further consideration of the Scully-Lamb laser model, which provides a baseline for more recent work on nonlinear optical systems.

The following two sections discuss the monitoring of fluctuations in quantum and classical populations respectively. In section 4 analysis of the Scully-Lamb laser model shows that the consistent adoption of quantum or classical monitoring schemes leads to identical estimates for the population fluctuation time. A summary and conclusions are presented in section 5.

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2. Quantum population correlations

Consider a population of quantum boson particles that shows statistical fluctuations in the number of individuals. If the fluctuations are caused by stationary processes, the correlation between the numbers of individuals at times separated by a period τ may be denoted

$$G_{\mathrm{q}}^{(2)}(\tau)=\langle\hat{a}^{\dagger}(0)\hat{a}^{\dagger}(\tau)\hat{a}(\tau)\hat{a}(0)\rangle \tag{2.1}$$

where the ordering of creation and destruction operators at times 0 and τ is appropriate to a measurement that is accomplished by a joint absorption of particles at the two instants of time (Glauber 1963). If $P_n(0)$ is the equilibrium statistical distribution of the particles amongst the number states $|n\rangle$ and there is no dynamical coupling of these diagonal elements of the density matrix to off-diagonal elements, the correlation (2.1) can be written

$$G_{\mathrm{q}}^{(2)}(\tau)=\sum_n\langle n-1|\hat{a}^{\dagger}(\tau)\hat{a}(\tau)|n-1\rangle nP_n(0) \tag{2.2}$$

where the usual properties of the boson operators have been used. Let $P(m, \tau|n-1, 0)$ be the conditional probability that there are m particles at time τ given that there were $n-1$ particles at time 0 The correlation (2.2) thus becomes

$$G_{\mathrm{q}}^{(2)}(\tau)=\sum_n\sum_m mP(m,\tau|n-1,0)nP_n(0). \tag{2.3}$$

The correlation for $\tau=0$ takes the simple form

$$G_{\mathrm{q}}^{(2)}(0)=\sum_n(n-1)nP_n(0)=\langle n(n-1)\rangle. \tag{2.4}$$

The correlation for very long times t also takes a simple form

$$G_{\mathrm{q}}^{(2)}(\infty)=\langle n\rangle^2 \tag{2.5}$$

when the statistical fluctuations of the population have finite correlation times. If only a single correlation time is important, the results (2.4) and (2.5) for the two ends of the range of τ can be interpolated by the form

$$G_{\mathrm{q}}^{(2)}(\tau)=\langle n\rangle^2+\{\langle n(n-1)\rangle-\langle n\rangle^2\}\exp(-\lambda_{\mathrm{q}}|\tau|) \tag{2.6}$$

The inverse correlation time λ_{q} is determined by comparison of the time derivatives of (2.3) and (2.6),

$$\begin{aligned}\left.\frac{\mathrm{d}G_{\mathrm{q}}^{(2)}(\tau)}{\mathrm{d}\tau}\right|_{\tau=0}&=\sum_n\sum_m m\left.\frac{\mathrm{d}P(m,\tau|n-1,0)}{\mathrm{d}\tau}nP_n(0)\right|_{\tau=0}\\&=-\lambda_{\mathrm{q}}\{\langle n(n-1)\rangle-\langle n\rangle^2\}\end{aligned} \tag{2.7}$$

and explicit expressions for λ_{q} can be found for processes with known time derivatives of the conditional probability elements. Such expressions are of course exact for systems that have a single exponential decay process, but useful approximations can also be found for systems with more complicated dynamics, and this is illustrated by the laser photon population correlation treated in section 4. A relation equivalent to (2.7) has previously been given by Hildred and Hall (1978) following Jakeman and Pike (1971).

Consider now an experiment in which particles are detected by removal from the population at an individual rate γ. For the example of a population of photons inside a single-ended optical cavity with a detector placed outside the cavity, γ is the photon transmission rate through the cavity mirror. The flow of particles into the detector can be described in terms of destruction operators

$$\hat{b}(t) = \gamma^{1/2}\hat{a}(t) - \hat{a}_{in}(t) \tag{2.8}$$

where $\hat{a}_{in}(t)$ is associated with the flow of particles back into the population (Collett and Gardiner 1984). Such a reverse flow is here assumed to be absent, with the excitation described by $\hat{a}_{in}(t)$ taken to be in its vacuum state The spectrum of the population fluctuations is defined to be (see for example Haake *et al* 1989)

$$S(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\tau \exp(i\omega\tau)\{\langle\hat{b}^\dagger(0)\hat{b}(0)\hat{b}^\dagger(\tau)\hat{b}(\tau)\rangle - \langle\hat{b}^\dagger(0)\hat{b}(0)\rangle^2\} \tag{2.9}$$

Then use of the commutation property

$$[\hat{b}(0), \hat{b}^\dagger(\tau)] = \delta(\tau) \tag{2.10}$$

together with the definition (2.1), gives

$$S(\omega) = \frac{1}{2\pi}\gamma\langle n\rangle + \frac{1}{2\pi}\gamma^2\int_{-\infty}^{\infty} d\tau \exp(i\omega\tau)\{G_q^{(2)}(\tau) - \langle n\rangle^2\} \tag{2.11}$$

Insertion of the form of correlation from (2.6) gives the spectrum

$$S(\omega) = \frac{\gamma\langle n\rangle}{2\pi} + \{\langle n(n-1)\rangle - \langle n\rangle^2\}\frac{\gamma^2\lambda_q/\pi}{\omega^2+\lambda_q^2} \tag{2.12}$$

where the first term on the right represents the detection shot noise.

3. Classical population correlations

It is interesting to compare the above analysis with that for a classical population of individuals. This will also help to resolve certain discrepancies between results in the literature on the subject (see section 4) An investigation of the relationship between measurements on classical and quantum populations has been carried out previously by Shepherd (1981), Jakeman and Shepherd (1984). The techniques developed in these papers have subsequently been applied to quantum mechanical problems (Shepherd and Jakeman 1987) and in the development of new classical population models (Jakeman 1990).

The correlations between numbers of individuals at times separated by a period τ in a classical population is given by

$$G_c^{(2)}(\tau) = \sum_n\sum_m mP(m, \tau | n, 0)nP_n(0) \tag{3.1}$$

where

$$P(m, \tau; n, 0) = P(m, \tau | n, 0)P_n(0) \tag{3.2}$$

expresses the usual relationships between the equilibrium, conditional and joint distributions of finding n individuals present in a classical population at time zero and m at time τ.

Equation (3.1) differs from the quantum mechanical expression (2.3) only through the appearance of n in the conditional distribution on the right-hand side rather than $(n-1)$. *In fact equation* (2.3) *is obtained for a classical population when counted* *individuals are removed so that the evolution of the process is perturbed by the measurement, or equivalently when the population is monitored through measurement of the number of individuals leaving the population in a small fixed time interval* (Jakeman and Shepherd 1984). It is worth noting here that photon counting experiments measure such a *flux* of events, i.e. the number of particles arriving at the detector in some sample time T. Only when this time is short compared with the fluctuation time of the light will the number correlation function (2 3) coincide with the experimentally measured quantity. The effect of finite integration time on measurements of the flux of individuals leaving a population has been reported by Shepherd (1981) and Shepherd and Jakeman (1987). This problem will not be considered further in the present paper.

Equation (3 1) implies that

$$G_c^{(2)}(0) = \langle n^2 \rangle \tag{3 3}$$

and

$$G_c^{(2)}(\infty) = \langle n \rangle^2 \tag{3 4}$$

so that if only a single correlation time is important, the correlation function of classical number fluctuations can be interpolated by the relation

$$G_c^{(2)}(\tau) = \langle n \rangle^2 + [\langle n^2 \rangle - \langle n \rangle^2] \exp(-\lambda_c |\tau|). \tag{3.5}$$

The characteristic time constant is given in terms of the zero delay derivative of this quantity through the formula

$$\left.\frac{\mathrm{d}G_c^{(2)}}{\mathrm{d}\tau}\right|_{\tau=0} = -\lambda_c[\langle n^2 \rangle - \langle n \rangle^2] = -\lambda_c \operatorname{Var}(n) \tag{3.6}$$

which may be compared with equation (2.7) for a quantum population. Substituting (3.1) into equation (3.6) leads to the following expression for the time constant

$$\lambda_c = -[\operatorname{Var}(n)]^{-1} \sum_n \sum_m nmP_n(0) \left.\frac{\mathrm{d}P(m, \tau | n, 0)}{\mathrm{d}\tau}\right|_{\tau=0}. \tag{3.7}$$

Spectral characterization of classical population processes does not feature prominently in the literature. However, the spectral density of a stationary point process is given by (Cox and Lewis 1966)

$$S(\omega) = \frac{1}{2\pi}\langle m \rangle + \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{d}\tau \exp(\mathrm{i}\omega\tau)\{\langle m(t)m(t+\tau)\rangle - \langle m \rangle^2\} \tag{3.8}$$

where $m(t)$ is the rate of change of the number of events counted in a short time interval centred at time t The form of (3 8) results from the addition of a term $\langle m \rangle \delta(\tau)$ to the covariance density of the differential process, $m(\tau)$, in order that the correct result be obtained at zero delay time Note that the structure of the equation is identical to that of (2.11) with $\langle m \rangle = \gamma \langle n \rangle$ and $\langle m(0)m(\tau) \rangle = \gamma^2 G_q^{(2)}(\tau)$, and confirms the interpretation of (2.3) as a flux measurement valid for integration times short compared to the characteristic fluctuation time of the event train

4. Application to the Scully–Lamb laser model

The Scully–Lamb model is valid for a laser in which the relevant atomic transition rates are at least an order of magnitude greater than the rate of loss of photons from the laser cavity The photon-number probability distribution then satisfies the equation of motion (Scully and Lamb 1967, Loudon 1983)

$$\frac{dP_n}{dt} = -\frac{\alpha n_s(n+1)}{n_s+n+1} P_n + \frac{\alpha n_s n}{n_s+n} P_{n-1} - \gamma n P_n + \gamma(n+1)P_{n+1} \tag{4.1}$$

where α is the scaled pumping rate and γ is the photon loss rate through one mirror of the laser cavity, the other mirror having 100% reflectivity The laser threshold occurs at $\alpha = \gamma$, and the 'saturation' photon number n_s is the mean of the distribution at $\alpha = 2\gamma$.

The rates of change of the conditional probabilities needed for application of (2.7) now follow directly from (4.1) For each value of n in (2.7), the derivatives are nonzero only for $m = n$, $n-1$ and $n-2$, and these are obtained by setting n on the left of (4.1) equal to the three values of m in turn and retaining only the contributions on the right that survive when $P_{n-1} = 1$ and the other elements vanish The solution of (2.7) is found after some algebra to be

$$\lambda_q = \frac{\gamma}{\langle n(n-1)\rangle - \langle n\rangle^2}\left(n_s \sum_{n=0}^{\infty} \frac{n-1}{n_s+n-1} P_n + P_0\right). \tag{4.2}$$

It is best to consider separately the three regions of laser operation.

4.1. *Below threshold* ($\alpha < \gamma$)

Standard results given in the above references are

$$\langle n\rangle = \alpha/(\gamma-\alpha) \tag{4.3}$$

$$\langle n(n-1)\rangle = 2\langle n\rangle^2 \tag{4.4}$$

and

$$P_0 = 1-(\alpha/\gamma). \tag{4.5}$$

The saturation number n_s is typically much larger than any n for which the below-threshold distribution P_n has significant values Thus $n-1$ can be neglected in the denominator of the summand in (4.2), and

$$\lambda_q = \gamma - \alpha \tag{4.6}$$

while the population correlation is

$$G_q^{(2)}(\tau) = \langle n\rangle^2\{1+\exp[-(\gamma-\alpha)|\tau|]\} \tag{4.7}$$

in agreement with known results for a chaotic photon distribution (Hildred and Hall 1978, Loudon 1983) but in disagreement with Jakeman and Pike (1971). The spectrum obtained from (2.12) is

$$S(\omega) = \frac{\gamma\langle n\rangle}{2\pi} + \frac{\gamma^2\langle n\rangle^2\lambda_q/\pi}{\omega^2+\lambda_q^2} \tag{4.8}$$

where $\langle n\rangle$ is given by (4.3) and λ_q by (4.6), and this applies to observations outside the laser cavity of photons transmitted through the partially reflecting mirror. The final term in (4.8) arises from self-beating of chaotic amplitude fluctuations whose Lorentzian spectrum has a full width λ_q at half maximum height.

The inverse correlation times predicted by (4 6) and illustrated in figure 1 can be compared with the results obtained by numerical solution of the Scully–Lamb equations of motion (Smith 1975) shown in the inset in the figure The validity of (4 6) is restricted to operation of the laser outside its threshold region, which extends over values of α/γ within a few times $n_s^{1/2}$ of the value unity. This region is relatively broad for the low value $n_s = 1600$ assumed by Smith for the results given in his table I. However, with the threshold region excluded, the approximation (4.6) gives inverse correlation times that agree with the exact values to within about 10% or less.

4.2. *At threshold* ($\alpha = \gamma$)

The standard results of the Scully–Lamb model are

$$\langle n\rangle = (2n_s/\pi)^{1/2} \tag{4.9}$$

$$\langle n(n-1)\rangle = \tfrac{1}{2}\pi\{1-(2/\pi n_s)^{1/2}\}\langle n\rangle^2 \tag{4.10}$$

and

$$P_0 = (2/\pi n_s)^{1/2} \tag{4 11}$$

The saturation photon number n_s is usually much larger than unity, when

$$\lambda_q = \frac{\gamma}{\pi/2-1}\left(\frac{\pi}{2n_s}\right)^{1/2} = 2.20\frac{\gamma}{n_s^{1/2}} \tag{4.12}$$

to a good approximation. The results obtained from this expression are in better than 10% agreement with correlation times for several values of n_s calculated by numerical solution of the Scully–Lamb equation of motion (Smith 1975, table II). Thus for $n_s = 1600$, evaluation of (4.12) gives $\lambda_q/\gamma = 0.055$ compared to Smith's value of 0.052

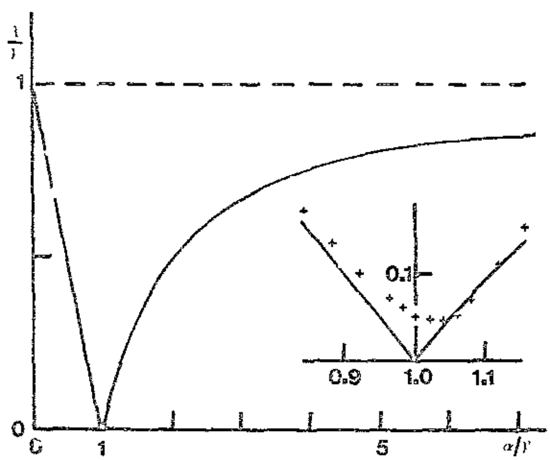

Figure 1. Variation with pumping rate of the inverse correlation time of a Scully–Lamb laser The continuous curves show the approximate results (4 6) and (4 15). The inset shows the threshold region on a larger scale with the exact numerical results of Smith (1975) for $n_s = 1600$ indicated by + symbols

4.3 Above threshold ($\alpha > \gamma$)

The standard results are

$$\langle n\rangle = (\alpha - \gamma)n_s/\gamma \tag{4.13}$$

$$\langle n(n-1)\rangle = \langle n\rangle^2 + n_s \tag{4.14}$$

and P_0 is negligibly small. The distribution P_n is now very sharply peaked around the large mean value $\langle n\rangle$ so that n can be replaced by $\langle n\rangle$ in the summand in (4.2) to give

$$\lambda_q = \frac{\gamma\langle n\rangle}{n_s + \langle n\rangle} = \frac{(\alpha-\gamma)\gamma}{\alpha} \tag{4.15}$$

and this relation is illustrated in figure 1. The population correlation is

$$G_q^{(2)}(\tau) = \langle n\rangle^2 + n_s \exp[-(\alpha-\gamma)\gamma|\tau|/\alpha] \tag{4.16}$$

and the external fluctuation spectrum is

$$S(\omega) = \frac{\gamma\langle n\rangle}{2\pi} + \frac{\gamma^2 n_s \lambda_q/\pi}{\omega^2 + \lambda_q^2} \tag{4.17}$$

The final term in (4.17) can be interpreted as arising from beating of the coherent laser emission with residual amplitude fluctuations whose Lorentzian spectrum has a full width $2\lambda_q$ Note the additional factor of 2 compared to the width of the below-threshold amplitude fluctuation spectrum (Risken 1965).

These expressions disagree with the results of Jakeman and Pike (1971) and Hildred and Hall (1978). The spectrum (4 17) agrees in general with that of Haake *et al* (1989), but it only agrees with an expression quoted without derivation by Kennedy and Walls (1989) well above threshold, where $\alpha \gg \gamma$. Both of these papers derive an inverse correlation time that agrees with the expression given in (4.15), and there is again good agreement with the exact results of Smith (1975), provided that the threshold region is excluded from the comparison.

Some of these discrepancies can be resolved by repeating the above calculations without photon annihilation. Thus we assume that the rate equation (4.1) characterizes a classical population of individuals and use result (3.7) to evaluate its characteristic fluctuation time. This obtains the rather simple result

$$\lambda_c = \gamma\langle n\rangle/\mathrm{Var}n = \gamma/F = \gamma/(Q+1) \tag{4.18}$$

where F is the population Fano factor (Fano 1947) and Q is Mandel's parameter (Mandel 1979). Substituting the standard results (4.3) and (4 4) into equation (4.18) gives below threshold exactly

$$\lambda_c = \gamma - \alpha \qquad (\alpha \le \gamma). \tag{4.19}$$

Similarly, at threshold the standard results (4.9) and (4.10) lead through (4 18) without further approximation to

$$\lambda_c = \frac{\gamma}{\pi/2-1}\left(\frac{\pi}{2n_s}\right)^{1/2} \qquad (\alpha = \gamma) \tag{4.20}$$

whilst above threshold, substituting equations (4.13) and (4.14) into formula (4.18) gives exactly

$$\lambda_c = (\alpha-\gamma)\gamma/\alpha \qquad (\alpha > \gamma) \tag{4.21}$$

Evidently formulae (4.2) and (4.18) differ very little under the conditions when (4.6), (4.12) and (4.15) are valid. This is because below threshold the number fluctuations are characterized by a single exponential so that (2.6) and (3.5) are exact and $\lambda_q = \lambda_c$. At and above threshold the population is large so that its evolution is little affected by the removal of individuals during the counting process. Result (4.18) predicts a simple relationship between the magnitude and the timescale of the population fluctuating which should be open to experimental verification.

Comparison of the above results with those of Jakeman and Pike (1971) shows that these authors assumed the correct interpolation form (2.6) for the quantum mechanical correlation function but used the classical definition, $G_c^{(2)}(\tau)$ given by equation (3.1). Thus their formula for λ differs from both the quantum mechanical result (4.2) and the classical result (4 18). Whilst their result is accurate near and above threshold, assuming a large mean photon number, they fail to obtain the correct asymptotic limit below threshold where this number becomes small Hildred and Hall (1978) use the correct quantum mechanical starting formulae both for the interpolation form and the basic definition of $G^2(\tau)$. However, they fail to obtain the correct quantum mechanical result (4.2) but rather quote that obtained by Jakeman and Pike previously.

5. Conclusions

The main aim of this paper is the re-establishment of the method of Jakeman and Pike (1971) as a simple but effective procedure for determining good approximations to the time dependences of the particle number–number correlation functions of fluctuating boson populations We have developed general expressions for the correlation functions under the assumptions of a single characteristic correlation time and a counting mechanism that either destroys the counted particles (quantum version) or leaves them intact (classical version).

The general expressions have been illustrated by the example of a cavity laser treated by the theory of Scully and Lamb (1967). The quantum and classical versions produce different general expressions for the correlation function of the photon numbers at different times, but identical approximate results are obtained in the two versions when conditions appropriate to the three regimes of operation below, at and above threshold are inserted Below threshold the laser light has a chaotic character, when it is known that the quantum and classical calculations of the photon-number fluctuations give identical results since the additional 'particle' term in the quantum variance is compensated by the occurrence of $\langle n(n-1)\rangle$ instead of the classical $\langle n^2\rangle$ in the correlation function calculation. These two averages are again approximately the same at and above threshold on account of the large mean photon numbers that occur. Despite the limitation of the Jakeman and Pike method to a single correlation time, the laser results are close to those of a more accurate calculation that takes account of multiple relaxation time behaviour (Smith 1975).

More generally, we believe that the method herein corrected and developed in consistent quantum mechanical and classical versions offers a very useful technique for obtaining particle number correlation functions

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